

Eccentricity evolution in hierarchical triple systems with eccentric outer binaries

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ABSTRACT

We develop a technique for estimating the inner eccentricity in hierarchical triple systems, with the inner orbit being initially circular, while the outer one is eccentric. We consider coplanar systems with well separated components and comparable masses. The derivation of short period terms is based on an expansion of the rate of change of the Runge-Lenz vector. Then, the short period terms are combined with secular terms, obtained by means of canonical perturbation theory. The validity of the theoretical equations is tested by numerical integrations of the full equations of motion.

Key words: Celestial mechanics, stellar dynamics, binaries:general.

1 INTRODUCTION

A hierarchical triple system consists of a binary system and a third body on a wider orbit. The motion of such a system can be pictured as the motion of two binaries: the binary itself (inner binary) and the binary which consists of the third body and the centre of mass of the binary (outer binary). Hierarchical triple systems are widely present in the galactic field and in star clusters and studying the dynamical evolution of such systems is a key to understanding a number of issues in astronomy and astrophysics. Sometimes, for example, the inner pairs in triple stellar systems are *close* binary systems, i.e. the separation between the components is comparable to the radii of the bodies. In these circumstances, the behaviour of the inner binary can depend very sensitively on the separation of its components and this in turn is affected by the third body. Thus, a slight change in the separation of the binary stars can cause drastic changes in processes such as tidal friction and dissipation, mass transfer and mass loss due to a stellar wind, which may result in changes in stellar structure and evolution (e.g. Kiseleva, Eggleton & Mikkola 1998). Eventually, these physical changes can affect the dynamics of the whole triple system. But even in systems with well-separated inner binary components, the perturbation of the third body can have a devastating effect on the triple system as a whole (e.g. disruption of the system).

For most hierarchical triple stars, the period ratio X is of the order of 100 and these systems are probably very stable dynamically. However, there are systems with much smaller period ratios, like the system HD 109648 with $X = 22$ (Jha et al. 2000), the λ Tau system, with $X = 8.3$ (Fekel & Tomkin 1982) and the CH Cyg system with $X = 7.0$ (Hinkle et al. 1993).

In a previous paper (Georgakarakos 2002) we derived a formula for the inner eccentricity in hierarchical triple systems with coplanar and initially circular orbits. Now, the calculation is extended to systems with eccentric outer binaries (the inner orbit is still considered to be initially circular).

2 THEORY

We derive expressions for the short period and secular evolution of the inner eccentricity. Both short period and secular terms will be obtained as previously (Georgakarakos 2002), i.e. by using the definition of the Runge-Lenz vector for the former and by means of canonical perturbation theory for the latter. Again, at any moment of the evolution of the system, the eccentricity is considered to consist of a short period and a long period (secular) component, i.e. $e = e_{\text{short}} + e_{\text{sec}}$ (one can picture this by recalling the expansion of the disturbing function in solar system dynamics, where the perturbing potential is given as a sum of an infinite number of cosines of various frequencies). Thus, the eccentricity being initially zero implies that $e_{\text{short}} = -e_{\text{sec}}$ (initially).

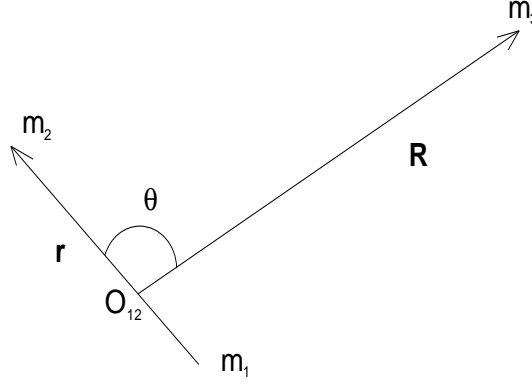


Figure 1. The Jacobi formulation. The point O_{12} is the centre of mass of the inner binary.

Finally, in this paper, we will be concentrating on systems with well separated components and comparable masses. Therefore, while developing the theoretical model in the next sections, we will consider X to be large (or any equivalent form of that assumption).

2.1 Calculation of the short-period contribution to the eccentricity

First, we calculate the short-period terms. The motion of the system can be studied using the Jacobi decomposition of the three-body problem (Fig. 1). In that context, the equation of motion of the inner binary is:

$$\ddot{\mathbf{r}} = -G(m_1 + m_2) \frac{\mathbf{r}}{r^3} + \mathbf{F}, \quad (1)$$

where \mathbf{F} , the perturbation to the inner binary motion, is

$$\mathbf{F} = Gm_3 \left(\frac{\mathbf{R} - \mu_1 \mathbf{r}}{|\mathbf{R} - \mu_1 \mathbf{r}|^3} - \frac{\mathbf{R} + \mu_2 \mathbf{r}}{|\mathbf{R} + \mu_2 \mathbf{r}|^3} \right) = Gm_3 \frac{\partial}{\partial \mathbf{r}} \left(\frac{1}{\mu_1 |\mathbf{R} - \mu_1 \mathbf{r}|} + \frac{1}{\mu_2 |\mathbf{R} + \mu_2 \mathbf{r}|} \right) \quad (2)$$

with

$$\mu_i = \frac{m_i}{m_1 + m_2}, \quad i = 1, 2.$$

Now, since the third star is at considerable distance from the inner binary, implying that r/R is small, the inverse distances in equation (2) can be expressed as:

$$\frac{1}{|\mathbf{R} - \mu_1 \mathbf{r}|} = \frac{1}{R} \sum_{n=0}^{\infty} \left(\frac{\mu_1 r}{R} \right)^n P_n(\cos \theta)$$

and

$$\frac{1}{|\mathbf{R} + \mu_2 \mathbf{r}|} = \frac{1}{R} \sum_{n=0}^{\infty} \left(-\frac{\mu_2 r}{R} \right)^n P_n(\cos \theta),$$

where P_n are the Legendre polynomials and θ is the angle between the vectors \mathbf{r} and \mathbf{R} . Expanding to third order, the perturbation becomes

$$\mathbf{F} = Gm_3 \frac{\partial}{\partial \mathbf{r}} \left(\frac{3}{2} \frac{(\mathbf{r} \cdot \mathbf{R})^2}{R^5} - \frac{1}{2} \frac{r^2}{R^3} - \frac{5(\mu_2^2 - \mu_1^2)}{2} \frac{(\mathbf{r} \cdot \mathbf{R})^3}{R^7} + \frac{3(\mu_2^2 - \mu_1^2)}{2} \frac{r^2 (\mathbf{r} \cdot \mathbf{R})}{R^5} \right). \quad (3)$$

The first two terms in the above equation come from the quadrupole term (P_2), while the other two come from the octupole term (P_3).

Using now the definition of the eccentric vector, i.e. the vector which has the same direction as the radius vector to the pericentre and whose magnitude is equal to the eccentricity of the orbit, we can obtain an expression for the inner eccentricity. The inner eccentric vector \mathbf{e}_1 is given by

$$\mathbf{e}_1 = -\frac{\mathbf{r}}{r} + \frac{1}{\mu} (\dot{\mathbf{r}} \times \mathbf{h}), \quad (4)$$

where $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ and $\mu = G(m_1 + m_2)$. Differentiating equation (4) and substituting for \mathbf{F} (we neglect the term $\mathbf{r} \cdot \dot{\mathbf{r}}$ because, for the applications discussed in this paper, is expected to be small and of $O(e)$), we obtain:

$$\begin{aligned} \dot{\mathbf{e}}_1 = & \frac{Gm_3}{\mu R^3} \left[\left(6 \frac{(\mathbf{r} \cdot \mathbf{R})(\dot{\mathbf{r}} \cdot \mathbf{R})}{R^2} - 15(\mu_2^2 - \mu_1^2) \frac{(\mathbf{r} \cdot \mathbf{R})^2(\dot{\mathbf{r}} \cdot \mathbf{R})}{R^4} + 3(\mu_2^2 - \mu_1^2) \frac{r^2(\dot{\mathbf{r}} \cdot \mathbf{R})}{R^2} \right) \mathbf{r} + \right. \\ & \left. + \left(r^2 - 3 \frac{(\mathbf{r} \cdot \mathbf{R})^2}{R^2} + \frac{15}{2}(\mu_2^2 - \mu_1^2) \frac{(\mathbf{r} \cdot \mathbf{R})^3}{R^4} - \frac{9}{2}(\mu_2^2 - \mu_1^2) \frac{r^2(\mathbf{r} \cdot \mathbf{R})}{R^2} \right) \dot{\mathbf{r}} \right]. \end{aligned} \quad (5)$$

Now, the Jacobi vectors can be represented approximately in polar form as $\mathbf{r} = a_1(\cos n_1 t, \sin n_1 t)$ and $\mathbf{R} = R(\cos(f + \varpi), \sin(f + \varpi))$, where a_1 is the semi-major axis of the inner orbit, n_1 is the mean motion of the inner orbit, $R = \frac{a_2(1-e^2)}{1+e \cos f}$, a_2 the outer semi-major axis, e the outer eccentricity, and f and ϖ are the true anomaly and longitude of pericentre of the outer orbit. After integrating, the components x_1 and y_1 of the eccentric vector become (expanding in powers of $\frac{1}{X}$ and retaining the four leading terms):

$$x_1(t) = \frac{m_3}{M} \frac{1}{X^2(1-e^2)^3} \left(P_{x21}(t) + \frac{1}{X} P_{x22}(t) + m_* X^{\frac{1}{3}} P_{x31}(t) + m_* \frac{1}{X^{\frac{2}{3}}} P_{x32}(t) \right) + C_{x_1} \quad (6)$$

$$y_1(t) = \frac{m_3}{M} \frac{1}{X^2(1-e^2)^3} \left(P_{y21}(t) + \frac{1}{X} P_{y22}(t) + m_* X^{\frac{1}{3}} P_{y31}(t) + m_* \frac{1}{X^{\frac{2}{3}}} P_{y32}(t) \right) + C_{y_1} \quad (7)$$

where

$$P_{x21}(t) = (1 + e \cos f)^3 \left[-\frac{1}{2} \cos n_1 t + \frac{1}{4} \cos(3n_1 t - 2f - 2\varpi) + \frac{9}{4} \cos(n_1 t - 2f - 2\varpi) \right] \quad (8)$$

$$\begin{aligned} P_{x22}(t) = & \frac{(1 + e \cos f)^4}{(1 - e^2)^{\frac{3}{2}}} \left[\frac{9}{2} \cos(n_1 t - 2f - 2\varpi) + \frac{1}{6} \cos(3n_1 t - 2f - 2\varpi) + e \left[-\frac{3}{4} \cos(n_1 t - f) + \frac{3}{4} \cos(n_1 t + f) + \right. \right. \\ & \left. \left. + \frac{45}{8} \cos(n_1 t - 3f - 2\varpi) + \frac{5}{24} \cos(3n_1 t - 3f - 2\varpi) - \frac{9}{8} \cos(n_1 t - f - 2\varpi) - \frac{1}{24} \cos(3n_1 t - f - 2\varpi) \right] \right] \end{aligned} \quad (9)$$

$$\begin{aligned} P_{x31}(t) = & (1 - e^2)^{\frac{1}{2}} \left[\frac{15}{16} \cos(f + \varpi) + \frac{15}{32} e \cos(2f + \varpi) + e^2 \left[\frac{45}{32} \cos(f + \varpi) - \frac{75}{64} \cos(f - \varpi) + \frac{5}{64} \cos(3f + \varpi) \right] + \right. \\ & \left. + e^3 \left[\frac{45}{128} \cos(2f - \varpi) - \frac{45}{128} \cos(2f + \varpi) \right] + e^4 \left[\frac{5}{32} \cos(3f + \varpi) - \frac{5}{32} \cos(3f - \varpi) \right] \right] \end{aligned} \quad (10)$$

$$P_{x32}(t) = \frac{(1 + e \cos f)^4}{(1 - e^2)} \left[\frac{3}{32} \cos(2n_1 t - f - \varpi) - \frac{45}{32} \cos(2n_1 t - 3f - 3\varpi) - \frac{15}{64} \cos(4n_1 t - 3f - 3\varpi) \right] \quad (11)$$

$$P_{y21}(t) = (1 + e \cos f)^3 \left[-\frac{1}{2} \sin n_1 t + \frac{1}{4} \sin(3n_1 t - 2f - 2\varpi) - \frac{9}{4} \sin(n_1 t - 2f - 2\varpi) \right] \quad (12)$$

$$\begin{aligned} P_{y22}(t) = & \frac{(1 + e \cos f)^4}{(1 - e^2)^{\frac{3}{2}}} \left[-\frac{9}{2} \sin(n_1 t - 2f - 2\varpi) + \frac{1}{6} \sin(3n_1 t - 2f - 2\varpi) + e \left[-\frac{3}{4} \sin(n_1 t - f) + \frac{3}{4} \sin(n_1 t + f) - \right. \right. \\ & \left. \left. - \frac{45}{8} \sin(n_1 t - 3f - 2\varpi) + \frac{5}{24} \sin(3n_1 t - 3f - 2\varpi) + \frac{9}{8} \sin(n_1 t - f - 2\varpi) - \frac{1}{24} \sin(3n_1 t - f - 2\varpi) \right] \right] \end{aligned} \quad (13)$$

$$\begin{aligned} P_{y31}(t) = & (1 - e^2)^{\frac{1}{2}} \left[\frac{15}{16} \sin(f + \varpi) + \frac{15}{32} e \sin(2f + \varpi) + e^2 \left[\frac{45}{32} \sin(f + \varpi) + \frac{75}{64} \sin(f - \varpi) + \frac{5}{64} \sin(3f + \varpi) \right] + \right. \\ & \left. + e^3 \left[-\frac{45}{128} \sin(2f - \varpi) - \frac{45}{128} \sin(2f + \varpi) \right] + e^4 \left[\frac{5}{32} \sin(3f + \varpi) + \frac{5}{32} \sin(3f - \varpi) \right] \right] \end{aligned} \quad (14)$$

$$P_{y32}(t) = \frac{(1 + e \cos f)^4}{(1 - e^2)} \left[\frac{3}{32} \sin(2n_1 t - f - \varpi) + \frac{45}{32} \sin(2n_1 t - 3f - 3\varpi) - \frac{15}{64} \sin(4n_1 t - 3f - 3\varpi) \right] \quad (15)$$

and

$$m_* = \frac{m_2 - m_1}{(m_1 + m_2)^{\frac{2}{3}} M^{\frac{1}{3}}}.$$

M is the total mass of the system and C_{x_1} and C_{y_1} are constants of integration. The semi-major axes, mean motions, outer eccentricity and longitude of pericentre were treated as constants in the above calculation. It should be mentioned here that in the expressions for $P_{x31}(t)$ and $P_{y31}(t)$ terms proportional to f appeared. To eliminate f from our expressions, we used the following series expansion (Murray & Dermott 1999)

$$f = l + 2e \sin f - \frac{3}{4} e^2 \sin 2f + \frac{1}{3} e^3 \sin 3f + O(e^4), \quad (16)$$

where l is the mean anomaly, i.e. we replaced f with the periodic part of the above equation.

2.2 Calculation of the secular contribution to the eccentricity

In order to derive the long-term modulation of the system, we use a Hamiltonian which is averaged over the inner and outer orbital periods by means of the Von Zeipel method. Secular terms cannot be obtained by the method of section 2.1, because, as we just saw, those terms appear as a linear function of time in the expansion of the eccentric vector and therefore, they are valid for limited time.

The doubly averaged Hamiltonian for coplanar orbits is (Marchal 1990):

$$H = -\frac{Gm_1m_2}{2a_S} - \frac{G(m_1+m_2)m_3}{2a_T} + Q_1 + Q_2 + Q_3, \quad (17)$$

where

$$Q_1 = -\frac{1}{8} \frac{Gm_1m_2m_3a_S^2}{(m_1+m_2)a_T^3(1-e_T^2)^{\frac{3}{2}}} (2+3e_S^2), \quad (18)$$

$$Q_2 = \frac{15Gm_1m_2m_3(m_1-m_2)a_S^3e_Se_T}{64(m_1+m_2)^2a_T^4(1-e_T^2)^{\frac{5}{2}}} \cos(g_S - g_T)(4+3e_S^2), \quad (19)$$

$$Q_3 = -\frac{15}{64} \frac{Gm_1m_2m_3^{\frac{7}{2}}a_S^2e_S^2(1-e_S^2)^{\frac{1}{2}}}{(m_1+m_2)^{\frac{3}{2}}M^{\frac{1}{2}}a_T^{\frac{9}{2}}(1-e_T^2)^3} [5(3+2e_T^2) + 3e_T^2 \cos 2(g_S - g_T)]. \quad (20)$$

The subscripts S and T refer to the inner and outer long period orbits respectively, while g is used to denote longitude of pericentre. The first term in the Hamiltonian is the Keplerian energy of the inner binary, the second term is the Keplerian energy of the outer binary, while the other three terms represent the interaction between the two binaries. The Q_1 term comes from the P_2 Legendre polynomial, the Q_2 term comes from the P_3 Legendre polynomial and the Q_3 term arises from the canonical transformation.

By using Hamilton's equations, we can now derive the averaged equations of motion of the system. Hence,

$$\begin{aligned} \frac{dx_S}{d\tau} &= \frac{5}{16} \alpha \frac{e_T}{(1-e_T^2)^{\frac{5}{2}}} (1-e_S^2)^{\frac{1}{2}} [(4+3e_S^2) \sin g_T + 6(x_S y_S \cos g_T + y_S^2 \sin g_T)] - \left[\frac{(1-e_S^2)^{\frac{1}{2}}}{(1-e_T^2)^{\frac{3}{2}}} + \frac{25}{8} \gamma \frac{3+2e_T^2}{(1-e_T^2)^3} (1 - \right. \\ &\quad \left. - \frac{3}{2} e_S^2) \right] y_S + \frac{15}{8} \gamma \frac{e_T^2}{(1-e_T^2)^3} [y_S \cos 2g_T - x_S \sin 2g_T - \frac{y_S}{2} (x_S^2 + 3y_S^2) \cos 2g_T + x_S (x_S^2 + 2y_S^2) \sin 2g_T] \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{dy_S}{d\tau} &= -\frac{5}{16} \alpha \frac{e_T}{(1-e_T^2)^{\frac{5}{2}}} (1-e_S^2)^{\frac{1}{2}} [(4+3e_S^2) \cos g_T + 6(x_S y_S \sin g_T + x_S^2 \cos g_T)] + \left[\frac{(1-e_S^2)^{\frac{1}{2}}}{(1-e_T^2)^{\frac{3}{2}}} + \frac{25}{8} \gamma \frac{3+2e_T^2}{(1-e_T^2)^3} (1 - \right. \\ &\quad \left. - \frac{3}{2} e_S^2) \right] x_S + \frac{15}{8} \gamma \frac{e_T^2}{(1-e_T^2)^3} [x_S \cos 2g_T + y_S \sin 2g_T - \frac{x_S}{2} (y_S^2 + 3x_S^2) \cos 2g_T - y_S (y_S^2 + 2x_S^2) \sin 2g_T] \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{dg_T}{d\tau} &= \frac{\beta(2+3e_S^2)}{2(1-e_T^2)^2} - \frac{5}{16} \frac{\alpha\beta(1+4e_T^2)}{e_T(1-e_T^2)^3} (4+3e_S^2) (x_S \cos g_T + y_S \sin g_T) + \frac{5}{8} \beta \gamma \frac{(1-e_S^2)^{\frac{1}{2}}}{(1-e_T^2)^{\frac{7}{2}}} [5e_S^2(11+4e_T^2) + 3(1+2e_T^2) \times \\ &\quad \times ((x_S^2 - y_S^2) \cos 2g_T + 2x_S y_S \sin 2g_T)] \end{aligned} \quad (23)$$

$$\frac{de_T}{d\tau} = \frac{5}{16} \frac{\alpha\beta}{(1-e_T^2)^2} (4+3e_S^2) (y_S \cos g_T - x_S \sin g_T) - \frac{15}{8} \beta \gamma \frac{e_T(1-e_S^2)^{\frac{1}{2}}}{(1-e_T^2)^{\frac{5}{2}}} (2x_S y_S \cos 2g_T - (x_S^2 - y_S^2) \sin 2g_T) \quad (24)$$

where

$$x_S = e_S \cos g_S, \quad y_S = e_S \sin g_S,$$

$$\alpha = \frac{m_1 - m_2}{m_1 + m_2} \frac{a_S}{a_T}, \quad \beta = \frac{m_1 m_2 M^{\frac{1}{2}}}{m_3 (m_1 + m_2)^{\frac{3}{2}}} \left(\frac{a_S}{a_T} \right)^{\frac{1}{2}}, \quad \gamma = \frac{m_3}{M^{\frac{1}{2}} (m_1 + m_2)^{\frac{1}{2}}} \left(\frac{a_S}{a_T} \right)^{\frac{3}{2}} \quad \text{and} \quad d\tau = \frac{3}{4} \frac{G^{\frac{1}{2}} m_3 a_S^{\frac{3}{2}}}{a_T^3 (m_1 + m_2)^{\frac{1}{2}}} dt.$$

After integrating the above averaged equations of motion for reasonable sets of parameters, using a 4th-order Runge-Kutta method with variable stepsize (Press et al. 1996), it was noticed that e_T remained almost constant. If that approximation is taken as an assumption, and terms of order e_S^2 are neglected and only the dominant term is retained in equation (23) (the dominant term is proportional to β , while the next order term is proportional to $\alpha\beta$, which, for the range of parameters discussed in this paper, is rather small compared to the dominant term), then the system can be reduced to one that can be solved analytically:

$$\begin{aligned} \frac{dx_S}{d\tau} &= -By_S + C \sin g_T \\ \frac{dy_S}{d\tau} &= Bx_S - C \cos g_T \\ \frac{dg_T}{d\tau} &= A, \end{aligned} \quad (25)$$

where

$$A = \frac{\beta}{(1 - e_T^2)^2}, \quad B = \frac{1}{(1 - e_T^2)^{\frac{3}{2}}} + \frac{25}{8} \gamma \frac{3 + 2e_T^2}{(1 - e_T^2)^3} \quad \text{and} \quad C = \frac{5}{4} \alpha \frac{e_T}{(1 - e_T^2)^{\frac{5}{2}}}.$$

The solution to system (25) is:

$$x_S(\tau) = C_1 \cos B\tau + C_2 \sin B\tau + \frac{C}{B - A} \cos(A\tau + g_{T_0}) \quad (26)$$

$$y_S(\tau) = C_1 \sin B\tau - C_2 \cos B\tau + \frac{C}{B - A} \sin(A\tau + g_{T_0}) \quad (27)$$

where C_1, C_2 are constants of integration and g_{T_0} is the initial value of g_T .

2.3 A formula for the inner eccentricity

In sections (2.1) and (2.2) we derived expressions for the short period and secular contribution to the inner eccentric vector. These can be combined to give an expression for the total eccentricity by replacing the constants in equations (6) and (7) by equations (26) and (27), since the latter evolve on a much larger timescale. This yields:

$$x_{in} = x_1(t) - C_{x_1} + x_S \quad (28)$$

$$y_{in} = y_1(t) - C_{y_1} + y_S \quad (29)$$

The constants C_1 and C_2 in equations (26) and (27) are determined by the fact that the inner eccentricity is initially zero and are found to be

$$C_1 = -x_1(0) - \frac{C}{B - A} \quad (30)$$

$$C_2 = y_1(0) + \frac{C}{B - A} \quad (31)$$

We are now able to obtain an expression for the inner eccentricity. Averaging over time and over the initial true anomaly f_0 and ϖ , the averaged square inner eccentricity will be given by:

$$\begin{aligned} \overline{e_{in}^2} = & \frac{m_3^2}{M^2} \frac{1}{X^4(1 - e^2)^{\frac{9}{2}}} \left[\frac{43}{8} + \frac{129}{8}e^2 + \frac{129}{64}e^4 + \frac{1}{(1 - e^2)^{\frac{3}{2}}} \left(\frac{43}{8} + \frac{645}{16}e^2 + \frac{1935}{64}e^4 + \frac{215}{128}e^6 \right) + \frac{1}{X^2(1 - e^2)^3} \left[\frac{365}{18} + \right. \right. \\ & + \frac{44327}{144}e^2 + \frac{119435}{192}e^4 + \frac{256105}{1152}e^6 + \frac{68335}{9216}e^8 + \frac{1}{(1 - e^2)^{\frac{3}{2}}} \left(\frac{365}{18} + \frac{7683}{16}e^2 + \frac{28231}{16}e^4 + \frac{295715}{192}e^6 + \frac{2415}{8}e^8 + \right. \\ & + \frac{12901}{2048}e^{10} \left. \right] + \frac{1}{X(1 - e^2)^{\frac{3}{2}}} \left[\frac{61}{3} + \frac{305}{2}e^2 + \frac{915}{8}e^4 + \frac{305}{48}e^6 + \frac{1}{(1 - e^2)^{\frac{3}{2}}} \left(\frac{61}{3} + \frac{854}{3}e^2 + \frac{2135}{4}e^4 + \frac{2135}{12}e^6 + \right. \right. \\ & + \frac{2135}{384}e^8 \left. \right] + m_*^2 X^{\frac{2}{3}} (1 - e^2) \left[\frac{225}{256} + \frac{3375}{1024}e^2 + \frac{7625}{2048}e^4 + \frac{29225}{8192}e^6 + \frac{48425}{16384}e^8 + \frac{825}{2048}e^{10} + \frac{1}{(1 - e^2)^{\frac{3}{2}}} \left(\frac{225}{256} + \right. \right. \\ & + \frac{2925}{1024}e^2 + \frac{775}{256}e^4 + \frac{2225}{8192}e^6 + \frac{25}{512}e^8 \left. \right] + m_*^2 \frac{1}{X^{\frac{4}{3}}(1 - e^2)^2} \left[\frac{8361}{4096} + \frac{125415}{8192}e^2 + \frac{376245}{32768}e^4 + \frac{41805}{65536}e^6 + \right. \\ & + \frac{1}{(1 - e^2)^{\frac{3}{2}}} \left(\frac{1575}{512} + \frac{11025}{256}e^2 + \frac{165375}{2048}e^4 + \frac{55125}{2048}e^6 + \frac{55125}{65536}e^8 \right) \left. \right] + 2 \left(\frac{C}{B - A} \right)^2. \end{aligned} \quad (32)$$

It should be mentioned here that in order to average the P_{31} term over f , we chose to expand $(1 + e \cos f)^{-2}$ binomially including terms up to $O(e^3)$. It should also be mentioned that numerical and theoretical tests showed that the total and secular outer eccentric vectors were initially almost equal and therefore in the evaluation of the above formula we consider $e_T = e$ and $g_{T_0} = \varpi$.

3 COMPARISON WITH NUMERICAL RESULTS

In order to test the validity of the formulae derived in the previous sections, we integrated the full equations of motion numerically, using a symplectic integrator with time transformation (Mikkola 1997).

The code calculates the relative position and velocity vectors of the two binaries at every time step. Then, by using standard two body formulae, we computed the orbital elements of the two binaries. The various parameters used by the code, were given the following values: writing index $Iwr = 1$, average number of steps per inner binary period $NS = 60$, method coefficients $a1 = 1$ and $a2 = 15$, correction index $icor = 1$. In all simulations, we confined ourselves to systems with mass ratios within the range $10 : 1$ since, among stellar triples, mass ratios are rare outside a range of approximately $10 : 1$, although

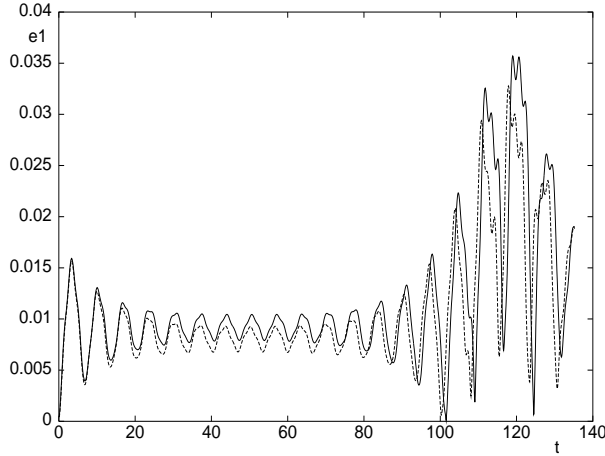


Figure 2. Inner eccentricity against time for a system with $m_1 = 0.3$, $m_3 = 3$, $e = 0.4$, $X_{0f} = 10$, $\varpi = 0^\circ$ and $f_0 = 90^\circ$. The integration time span is one outer orbital period ($T_{\text{out}} = 135.2$). The continuous curve comes from the numerical integration of the full equations of motion, while the dashed curve is a plot of equations (28) and (29). In the system of units used, the inner binary period is $T_{\text{in}} = 2\pi$.

such systems would be inherently difficult to recognise' (Eggleton & Kiseleva 1995); We also introduced the fictitious initial period ratio X_{0f} , defined as the ratio of the period that the outer binary would have on a circular orbit with a semi major axis equal to its periastron distance over the period of the inner binary. In all cases $X_{0f} \geq 10$. We also used units such that $G = 1$ and $m_1 + m_2 = 1$ and we always started the integrations with $a_1 = 1$. In that system of units, the initial conditions for the numerical integrations were as follows:

$$r_1 = 1, \quad r_2 = 0, \quad r_3 = 0$$

$$R_1 = R_0 \cos(f_0 + \varpi), \quad R_2 = R_0 \sin(f_0 + \varpi), \quad R_3 = 0$$

$$\dot{r}_1 = 0, \quad \dot{r}_2 = 1, \quad \dot{r}_3 = 0$$

$$\dot{R}_1 = -\sqrt{\frac{M}{a_2(1-e^2)}} \sin(f_0 + \varpi), \quad \dot{R}_2 = \sqrt{\frac{M}{a_2(1-e^2)}} \cos(f_0 + \varpi), \quad \dot{R}_3 = 0.$$

3.1 SHORT PERIOD EVOLUTION

First we tested the validity of equations (28) and (29) in the short term. The results are presented in Tables 1 and 2, which give the percentage error between the averaged, over time, numerical and theoretical e_{in} (the theoretical eccentricity was obtained by evaluating equations (28) and (29) everytime we had an output from the symplectic integrator; both averaged numerical and theoretical eccentricities were calculated by using the trapezium rule). Table 1 presents results for $e = 0.4$, while Table 2 presents results for $e = 0.75$. For each pair (m_3, X_0) in these Tables, there are five entries, corresponding, from top to bottom, to the following inner binaries: $m_1 = 0.1 - m_2 = 0.9$, $m_1 = 0.2 - m_2 = 0.8$, $m_1 = 0.3 - m_2 = 0.7$, $m_1 = 0.4 - m_2 = 0.6$ and $m_1 = 0.5 - m_2 = 0.5$. A dash denotes that the analogy among the masses was outside the range 10 : 1. The integrations were performed over one outer orbital period time span and were done for $f_0 = 90^\circ$ and $\varpi = 0^\circ$. However, this does not affect the qualitative understanding of the problem at all.

The results generally show a good agreement between the numerical and theoretical eccentricity. All the errors are below 10% and they drop as we move to larger values of X_{0f} . Similar results are obtained for different f_0 and ϖ . However the theory is not very accurate for highly eccentric outer binaries, because of the expansion for f given by equation (16). In this case, higher order terms with respect to the eccentricity should be included in equation (16) for a better approach to the problem. The masses of the inner binary also play an important role in that specific issue, as the P_3 term is proportional to $(m_1 - m_2)$ and therefore the closer we are in a situation where the inner binary has equal masses the smaller the problem of neglecting terms of $O(e^4)$ in equation (16) will be. Finally, the theory can work well even when our short period model includes only the P_{21} and P_{31} terms. The difference is more apparent for smaller X_{0f} and for smaller outer eccentricities where the full model works much better in those cases. It is also necessary when $m_1 = m_2$.

Fig. 2 is a plot of inner binary eccentricity against time for a system with $m_1 = 0.3$, $m_3 = 3$, $e = 0.4$, $X_{0f} = 10$, $\varpi = 0^\circ$ and $f_0 = 90^\circ$. The continuous curve has been produced as a result of the numerical integration of the full equations of motion, while the dashed curve is based on equations (28) and (29). The error for this case, as seen in Table 1, was 9.6%.

Table 1. Percentage error between the averaged numerical and averaged theoretical e_{in} . The theoretical model is based on equations (28) and (29). For all systems, $e = 0.4$, $f_0 = 90^\circ$ and $\varpi = 0^\circ$.

$m_3 \setminus X_0$	10	15	20	25	30	50
0.05	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
0.09	2.6	1.2	0.8	0.5	0.4	0.2
	3.5	1.9	1.3	1	0.8	0.5
	3.3	1.8	1.2	0.9	0.7	0.4
	3.2	1.6	1.1	0.8	0.6	0.3
	3	1.5	1	0.7	0.5	0.3
	2.8	1.4	0.9	0.6	0.5	0.3
0.5	6.5	4.1	3.1	2.4	2	1.3
	6	3.6	2.7	2	1.7	1.1
	5.5	3.2	2.3	1.7	1.4	0.9
	5.1	3	2.1	1.6	1.2	0.7
	4.9	2.8	2	1.4	1.1	0.6
1	8.4	5.3	4	3.3	2.7	1.7
	7.7	4.8	3.5	2.8	2.3	1.5
	7.1	4.3	3.1	2.4	2	1.2
	6.6	4	2.8	2.1	1.7	1
	6.4	3.8	2.7	1.9	1.5	0.9
1.5	-	-	-	-	-	-
	8.7	5.4	4.1	3.1	2.6	1.6
	8.1	5	3.7	2.7	2.3	1.3
	7.6	4.6	3.3	2.4	2	1.1
	7.3	4.4	3.1	2.2	1.8	1
2	-	-	-	-	-	-
	9.3	5.8	4.3	3.4	2.8	1.7
	8.7	5.3	3.9	3	2.4	1.4
	8.2	5	3.5	2.7	2.1	1.2
	7.8	4.8	3.4	2.5	2	1.1
2.6	-	-	-	-	-	-
	-	-	-	-	-	-
	9.3	5.6	4.2	3.1	2.5	1.5
	8.8	5.3	3.8	2.8	2.3	1.3
	8.4	5.2	3.7	2.6	2.1	1.2
3	-	-	-	-	-	-
	-	-	-	-	-	-
	9.6	5.8	4.2	3.2	2.5	1.5
	9	5.5	3.9	2.9	2.4	1.3
	8.7	5.3	3.8	2.8	2.2	1.2
3.4	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	9.3	5.6	4	3	2.4	1.4
	8.9	5.5	3.9	2.8	2.2	1.3
4	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	9.6	5.7	4.2	3.1	2.5	1.4
	9.3	5.6	4	2.9	2.4	1.3
4.5	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	9.5	5.7	4.1	3	2.3	1.3
5	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	9.7	5.8	4.2	3	2.4	1.4

Table 2. Percentage error between the averaged numerical and averaged theoretical e_{in} . The theoretical model is based on equations (28) and (29). For all systems, $e = 0.75$, $f_0 = 90^\circ$ and $\varpi = 0^\circ$.

$m_3 \setminus X_0$	10	15	20	25	30	50
0.05	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
0.09	0.8	0.4	0.3	0.2	0.1	0.1
	1.1	0.4	0.1	-0.2	-0.4	-1
	1.7	0.6	0.3	0.1	-0.1	-0.4
	1.8	0.7	0.4	0.3	0.2	-0.1
	1.5	0.6	0.4	0.3	0.2	0.1
0.5	1	0.5	0.4	0.3	0.2	0.1
	2.3	1.2	0.7	0.4	0.1	-0.5
	2.9	1.4	0.9	0.6	0.4	-0.1
	3	1.5	1	0.7	0.6	0.2
	2.9	1.5	1.1	0.8	0.6	0.3
1	2.7	1.5	1.1	0.9	0.7	0.4
	3.1	1.7	1.1	0.7	0.5	-0.1
	3.6	1.8	1.2	0.9	0.7	0.2
	3.7	2	1.4	1	0.8	0.4
	3.8	2.1	1.5	1.1	0.9	0.5
1.5	3.7	2.2	1.6	1.3	1	0.6
	-	-	-	-	-	-
	4	2.1	1.5	1.1	0.8	0.3
	4.2	2.3	1.6	1.2	1	0.5
	4.3	2.4	1.7	1.3	1.1	0.6
2	4.3	2.6	1.9	1.5	1.2	0.7
	-	-	-	-	-	-
	4.2	2.3	1.6	1.2	0.9	0.4
	4.5	2.5	1.8	1.4	1.1	0.6
	4.7	2.6	1.9	1.5	1.2	0.7
2.6	4.8	2.9	2.1	1.7	1.4	0.8
	-	-	-	-	-	-
	-	-	-	-	-	-
	4.8	2.7	1.9	1.5	1.2	0.6
	4.9	2.9	2.1	1.6	1.3	0.7
3	5.1	3.1	2.3	1.8	1.5	0.9
	-	-	-	-	-	-
	-	-	-	-	-	-
	4.9	2.8	2	1.5	1.2	0.7
	5.1	3	2.1	1.7	1.4	0.8
3.4	5.3	3.2	2.4	1.9	1.5	0.9
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	5.3	3	2.2	1.7	1.4	0.8
4	5.4	3.3	2.4	1.9	1.6	0.9
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	5.4	3.2	2.3	1.8	1.5	0.8
4.5	5.6	3.4	2.5	2	1.6	1
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
5	5.7	3.5	2.6	2	1.7	1
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	-	-	-	-	-	-
	5.9	3.6	2.6	2	1.7	1

Table 3. Percentage error between the averaged numerical e_{in}^2 and equation (32).

$m_3 \setminus X_{0f}$	10	20	30
0.09	-11.6	-2.5	-1
	106000	313000	625000
	22	2.3	1
	595000	1485000	2980000
0.5	-3.7	3.7	3.5
	7700	35500	82500
	-3.9	1	2
	40000	185000	427000
1	1	7.8	5.1
	4300	21000	51000
	-8.3	2.2	3
	23500	112000	265000
1.5	2	7.8	6.5
	3400	17000	41000
	-8.6	3.3	3.6
	18000	89000	214000
2	10.5	11.2	7.7
	2700	14500	36000
	-5.3	3.9	3.5
	15000	78000	190000

3.2 LONG PERIOD EVOLUTION

Next, we tested equation (32), which accounts for the short period and secular effects to the inner eccentricity. The formula was compared with results obtained from integrating the full equations of motion numerically. These results are presented in Table 3, which gives the percentage error between the averaged (over time, initial true anomaly f_0 and ϖ) numerical e_{in}^2 and equation (32). The error is accompanied by the period of the oscillation of the eccentricity, which is the same as the integration time span. There are four values per $(m_3 - X_{0f})$ pair. The first two (error-period) correspond to a system with $e = 0.4$, while the other two to a system with $e = 0.75$.

Each system was numerically integrated for $\varpi = 0^\circ - 360^\circ$ and $f_0 = 0^\circ - 360^\circ$ with a step of 45° . For a given value of ϖ and f_0 we integrated our system. After each run, e_{in}^2 was averaged over time using the trapezium rule and then we integrated the system for a different f_0 . After the integrations for all f_0 were done, we averaged over f_0 by using the rectangle rule. Then, the same procedure was applied for the next value of ϖ and when the integrations for all ϖ were done, we averaged over ϖ by using the rectangle rule. The integrations were also done for smaller steps in ϖ and f_0 , but there was not any difference in the outcome. All the integrations presented in Table 3 were done for $m_1 = 0.2$, but similar results are expected for the other inner binary masses (note that for $m_1 = m_2$ there is not any long period oscillation in the inner eccentricity, as seen in section 2.2).

Generally, it appears that the theory is in agreement with the numerical integrations. There are some cases where the eccentricity of the inner binary reaches significant values over a long period (e.g. for $X_{0f} = 10$, $m_3 = 0.09$ and $e = 0.75$ the inner maximum eccentricity is about 0.35) and therefore terms of $O(e_S^2)$ should be included in the secular equations. The contribution from short period terms becomes more noticeable and hence increasingly important as the outer eccentricity drops; and also as m_3 increases. Numerical integrations were also performed for 10 and 100 secular periods without any change in the error (something expected when the error in the rate of change of the eccentricity is quasi-periodic). Finally, we compared the results for the systems of Table 3 with a formula which only included the P_{21} and P_{31} terms. The maximum difference in the errors was around 3%.

4 DISCUSSION

We have derived a formula which gives an estimate for the inner eccentricity in hierarchical triple systems with eccentric outer binaries and coplanar orbits. The theoretical model appears to work satisfactory for the parameter ranges discussed. In cases with large outer eccentricities and significantly different inner binary masses, due to the approximation of $(1 + e \cos f)^{-2}$ with a series expansion in terms of e , the model could be inaccurate, especially for describing short term evolution. However, for hierarchical triple systems with highly eccentric outer binaries, it appears that the inner eccentricity is dominated by secular evolution, which is not affected by that approximation (the only effect of that approximation will be on determining the constants of integration C_1 and C_2 in section 2.3). A shorter formula can also be derived by only retaining the P_{21} and P_{31}

terms in equations (28) and (29). As the numerical integrations demonstrated in the previous section, the omission of those terms did not affect the situation very much. However, the contribution of the P_{22} and P_{32} terms is important in cases with significant short period evolution, i.e. small outer eccentricity and strong perturbation to the inner binary.

The theoretical results obtained in the previous sections could be applied in various astronomical and astrophysical topics. As it was stated in the introduction, the separation of the components in *close* binary systems can play a vital role in their evolution. For instance, the flow of material in a semidetached system can be seriously affected by a small change in the separation of the two stars and therefore it would be useful to have some information about the eccentricity injected into the binary by the third star. The theory could also be used in observational astronomy, to put constraints on different parameters of the observed systems, since there is always some uncertainty in their determination. For example, the theory could be used to rule whether the orbits of the triple system are coplanar or not, an interesting question which is related to the formation of the system.

The same technique can be applied to investigate systems with non coplanar orbits.

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